Radiation reaction in high-intensity fields

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Since the development of a radiating electron model by Dirac in 1938 [P. A. M. Dirac, Proc. R. Soc. Lond. A **167**, 148 (1938)], many authors have tried to reformulate this model of the so-called "radiation reaction". Recently, this effect has become important in ultra-intense laser–electron (plasma) interactions. In our recent research, we found a way of stabilizing the radiation reaction by quantum electrodynamics (QED) vacuum fluctuation [K Seto et al., Prog. Theor. Exp. Phys. **2014**, 043A01 (2014); K. Seto, Prog. Theor. Exp. Phys. **2015**, 023A01 (2015)]. On the other hand, the modification of the radiated field by highly intense incoming laser fields should be taken into account when the laser intensity is higher than 10^{22} W/cm^2, which could be achieved by next-generation ultra-short-pulse 10 PW lasers, like the ones under construction for the ELI–NP facility. In this paper, I propose a running charge–mass method for the description of the QED-based synchrotron radiation by high-intensity external fields with stabilization by the QED vacuum fluctuation as an extension from the model by Dirac.
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1. Introduction

With the rapid progress of ultra-short-pulse laser technology, the maximum intensities of these lasers have reached the order of 10^{22} W/cm^2 [1,2]. If the laser intensity is higher than this, strong radiation may be generated from a highly energetic electron. Accompanying this, the "radiation reaction", the feedback from radiation to an electron's motion, can have a strong influence on electrons in plasmas [3]. One of the facilities that can achieve these regimes, Extreme Light Infrastructure–Nuclear Physics (ELI–NP), will feature two 10 PW (approximately 10^{24} W/cm^2 at tightest focus) class lasers [4–6]. At these intensity levels, the radiation reaction must be taken into account in the laser–plasma experiments carried out. The original model of the radiation reaction, described by the Lorentz–Abraham–Dirac (LAD) equation [7], has a significant mathematical difficulty, an exponential divergence $dw/d\tau \propto \exp(\tau/\tau_0) \to \infty$, called "run-away" [7,8]. Here $\tau_0 = e^2/6\pi\varepsilon_0 m_0 c^3 = O(10^{-24}$ sec$)$, where m_0, e, and τ denote the rest mass, the charge, and the proper time of an electron. In my previous research, I succeeded in performing the stabilization of this run-away in the quantum electrodynamics (QED) vacuum fluctuation [8–10]. The last form of my equation was

$$\frac{dw^\mu}{d\tau} = -\frac{e}{m_0(1-\eta f_0)}\left[\mathfrak{F}^{\mu\nu} + \eta g_0(^*\mathfrak{F})^{\mu\nu}\right] w_\nu. \tag{1}$$

The vector space $\mathbb{V}_M{}^4$ denotes the set of vectors in Minkowski spacetime $(\mathbb{A}^4, g)$.[1] Defining $*\mathbb{V}_M{}^4$ as the dual space of $\mathbb{V}_M{}^4$, the Lorentz metric $g \in *\mathbb{V}_M{}^4 \otimes *\mathbb{V}_M{}^4$ has a signature of $(+, -, -, -)$, for $\forall a, b \in \mathbb{V}_M{}^4$, $g_{\mu\nu}a^\mu b^\nu = a^\nu b_\nu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \in \mathbb{R}$. w is the 4-velocity defined by $w = dx/d\tau = \gamma(c, \mathbf{v}) \in \mathbb{V}_M{}^4$. The field $\mathfrak{F} = F_{ex} + F_{LAD} \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4$, $F_{ex} \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4$ is an arbitrary external field, in our case generated by lasers. The field acting on an electron $F_{LAD} \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4$ is the radiation LAD field:

$$F_{LAD}{}^{\mu\nu}\big|_{x=x(\tau)} = -\frac{m_0 \tau_0}{ec^2} \left(\frac{d^2 w^\mu}{d\tau^2} w^\nu - w^\mu \frac{d^2 w^\nu}{d\tau^2} \right). \tag{2}$$

Since $\eta = O(\hbar^3)$, the limit $\hbar \to 0$ in Eq. (1) derives from the equation of motion $m_0 dw^\mu/d\tau = -e\mathfrak{F}^{\mu\nu} w_\nu$, the so-called LAD equation. f_0 and g_0 are Lorentz-invariant functions depending on the QED vacuum model. In the case of the Heisenberg–Euler vacuum [11,12], $f_0 = \langle \mathfrak{F}|\mathfrak{F} \rangle = \mathfrak{F}_{\mu\nu}\mathfrak{F}^{\mu\nu}$[2] and $g_0 = 7/4 \times \langle \mathfrak{F}|*\mathfrak{F} \rangle = 7/4 \times \mathfrak{F}_{\mu\nu}(*\mathfrak{F})^{\mu\nu}$ [8–10]. These works suggest that the QED vacuum fluctuation (i) stabilizes the LAD field and (ii) behaves well, since Eq. (1) agrees with one of the major references proposed by Landau and Lifshitz [10,13].

On the other hand, it is considered that the dynamics of an electron should be corrected in the highly intense fields produced by 10 PW lasers, by QED-based synchrotron radiation. In this physics regime, it is often discussed in terms of the parameter $\chi \in \mathbb{R}$, representing the field strength [14]:

$$\chi = \frac{3}{2} \frac{\lambda_C}{m_0 c^2} \sqrt{-g_{\mu\nu} f_{ex}{}^\mu f_{ex}{}^\nu}, \tag{3}$$

where the Compton length $\lambda_C = \hbar/m_0 c$. When one considers this in the rest frame, $\chi = 3/2 \times |\mathbf{E}_{ex}|_{rest}/E_{Schwinger}$. Here, $E_{Schwinger} = m_0{}^2 c^3/e\hbar$ is the critical field strength of light, namely the Schwinger limit. Therefore χ represents the external field strength or the intensity by using the ratio with this limit. By using QED-based synchrotron radiation with this χ dependence, I. Sokolov et al. [14] proposed the following radiation reaction model:

$$\frac{dp^\mu}{d\tau} = -eF_{ex}{}^{\mu\nu} \frac{dx_\nu}{d\tau} + \frac{\tau_0 q(\chi)}{m_0{}^2 c^2} g_{\alpha\beta} f_{ex}{}^\alpha f_{ex}{}^\beta p^\mu \tag{4}$$

$$\frac{dx^\mu}{d\tau} = \frac{1}{m_0} p^\mu + \tau_0 q(\chi) \frac{f_{ex}^\mu}{m_0}. \tag{5}$$

We will refer to this as the QED–Sokolov equation/model since the function $q(\chi)$ depends on the QED cross-section of synchrotron radiation with $r_\chi = r/(1 - \chi r)$:

$$q(\chi) = \frac{9\sqrt{3}}{8\pi} \int_0^{\chi^{-1}} dr\, r \left[\int_{r_\chi}^\infty dr'\, K_{5/3}(r') + \chi^2 r r_\chi K_{2/3}(r_\chi) \right].[3] \tag{6}$$

Equations (4)–(6) incorporate the modification of the QED radiation spectrum into the model [14]. In the low-intensity field regime, $\chi \ll 1$, then $q(\chi) \approx 1$. This limit converges to the result of the Landau–Lifshitz equation [13]. On the other hand, in the case of $\chi \sim 1$, which means a 10^{22} W/cm²-class laser and a GeV electron, $q(\chi) \sim 0.2$. So, the function $q(\chi)$ modifies radiation from the classical to quantum high-field dynamics. However, the QED–Sokolov equation violates the

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[1] $\mathbb{A}^4$ is the 4D affine space. The linear subspace of $\mathbb{A}^4$ should be $\mathbb{V}_M{}^4$.

[2] For $\forall A, B \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4$, $\langle A|B \rangle \equiv A_{\mu\nu}B^{\mu\nu}$.

[3] For $dW/dt|_{classic} = -\tau_0/m_0 \times g_{\alpha\beta} f_{ex}{}^\alpha f_{ex}{}^\beta$ as classical radiation energy loss (the Larmor formula), the QED-corrected formula becomes $dW/dt|_{High\ Field} = q(\chi) \times dW/dt|_{classic}$ [14].

Lorentz invariance, $(dx^{\mu}/d\tau)(dx_{\mu}/d\tau) = c^2$, which should be satisfied under classical dynamics; we should recover this requirement when we consider the classical-relativistic equation of motion.

It is natural to consider that the difference in the radiation field between classical dynamics and QED is the alteration of the source (current) term in Maxwell's equation. When we consider QED effects for the radiation reaction in the framework of classical dynamics, we need to insert a modulation of the charge–current density for describing the QED-based radiation field. In this paper, I discuss the general method of the model adaptation from the modified radiation field $F_{\text{Mod-LAD}}$ in high-intensity external fields (such as laser) to the field propagation in QED vacuum with new degrees of freedom such as the extension from Refs. [9] and [10]. By a combination of these, we can find the anisotropy of the coupling factor $\mathfrak{K} \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4$ between an electron and fields, which is a unique dynamical behavior predicted by this new model:

1) $F_{\text{LAD}} \mapsto F_{\text{Mod-LAD}}$ (Sect. 2.1)
2) QED vacuum (Sect. 2.2)

3) $\Xi = q(\chi)$
$\Rightarrow$
(Sect. 3.3/Appendices A)

$$\frac{dw^{\mu}}{d\tau} = -\frac{e}{m_0}\mathfrak{K}^{\mu\nu}{}_{\alpha\beta}(F_{\text{ex}}{}^{\alpha\beta} + F_{\text{Mod-LAD}}{}^{\alpha\beta})w_{\nu}.$$

To demonstrate this scheme, I will first introduce the new functions Ξ and Θ for the modification of the LAD field in the high-intensity external field. By using them, I will derive the modified-LAD field $F_{\text{Mod-LAD}}$ corresponding to the QED–Sokolov equation (Sect. 2.1) and correct the field $F_{\text{ex}} + F_{\text{Mod-LAD}}$ by QED vacuum fluctuation (Sect. 2.2). To simplify this, I perform it by field propagation in a Heisenberg–Euler vacuum in Sect. 3. We reach the conclusion that the new equation agrees well with the QED–Sokolov equation with the relation $(dx^{\mu}/d\tau)(dx_{\mu}/d\tau) = c^2$ and the anisotropic coupling between an electron and fields.

2. Modification by high-intensity field

In this section, I discuss the general method of how to treat the field $\mathfrak{F} \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4$ acting on an electron: $\mathfrak{F}^{\mu\nu} = \mathfrak{K}^{\mu\nu}{}_{\alpha\beta}\left(F_{\text{ex}}{}^{\alpha\beta} + F_{\text{Mod-LAD}}{}^{\alpha\beta}\right)$. By using this field, we can obtain the equation of motion of an electron.

2.1. Introduction of running charge and mass

In ultra-high-intensity fields, the coupling (charge) of an electron to fields may be modified due to the alteration of the current from classical dynamics to QED. This formulation has been discussed by I. Sokolov et al. and was introduced above as Eqs. (3)–(6) [14]. Finally, they formulated the following interesting relation:

$$\tau_0|_{\text{High Field}} = q(\chi) \times \tau_0 = q(\chi) \times \frac{e^2}{6\pi\varepsilon_0 m_0 c^3}, \tag{7}$$

where τ_0 is the constant in Eq. (2), $\tau_0 = e^2/6\pi\varepsilon_0 m_0 c^3 = O(10^{-24} \text{ sec})$, and $c\tau_0$ describes the order of the classical electron radius. Equation (7) suggests that the coupling e^2/m_0 should be replaced by $q(\chi) \times e^2/m_0$. It seems that Eq. (7) means the replacements of the charge $e \mapsto e' = e \times \sqrt{q(\chi)}$ and the LAD field $F_{\text{LAD}} \mapsto F'_{\text{LAD}} = \sqrt{q(\chi)} \times F_{\text{LAD}}$ since $F_{\text{LAD}} = O(e)$, but this is not correct. If we accept this replacement, $dw^{\mu}/d\tau = -e'/m_0 \times \left(F_{\text{ex}}{}^{\mu\nu} + F'_{\text{LAD}}{}^{\mu\nu}\right)w_{\nu} = -e/m_0 \times \left[\sqrt{q(\chi)} \times F_{\text{ex}}{}^{\mu\nu} + q(\chi) \times F_{\text{LAD}}{}^{\mu\nu}\right]w_{\nu}$ and the term of the external force $-\left[e \times \sqrt{q(\chi)}\right]/m_0 \times F_{\text{ex}}{}^{\mu\nu}w_{\nu}$ appear in the equation of motion. However, this term should be just $-e/m_0 \times F_{\text{ex}}{}^{\mu\nu}w_{\nu}$ for describing the incoming background field F_{ex} (see the QED–Sokolov equation (4)). In the case of making the replacements $e \mapsto e' = e \times q(\chi)$, $F_{\text{LAD}} \mapsto F'_{\text{LAD}} =$

$q(\chi) \times F_{\text{LAD}}$, and $m_0 \mapsto m_0{}' = m_0 \times q(\chi)$, it follows that $dw^\mu/d\tau = -e/m_0 \times [F_{\text{ex}}{}^{\mu\nu} + q(\chi) \times F_{\text{LAD}}{}^{\mu\nu}] w_\nu$, which is very similar to the form of the QED–Sokolov model in Eqs. (4)–(5). Therefore, this requires us to set the running charge and mass for the realization of QED-based synchrotron radiation like in the QED–Sokolov model.

Following the above idea, I pass to a more general discussion. The requirement for the modification of radiation is that the charge and mass of an electron should also be running. We introduce the new non-zero functions $\Xi, \Theta \in C^\infty(\mathbb{R})$ satisfying $q(\chi) = \Xi^2/\Theta$. Then, we can find the replacements of $e \mapsto e_{\text{High Field}} = e \times \Xi$ and $m_0 \mapsto m_{\text{High Field}} = m_0 \times \Theta$ with Eq. (7), $\tau_0 = e^2/6\pi\varepsilon_0 m_0 c^3 \mapsto \tau_0|_{\text{High Field}} = e_{\text{High Field}}{}^2/6\pi\varepsilon_0 m_{\text{High Field}} c^3$. The two functions Ξ and Θ should be the Lorentz invariants.

From here, we again try to derive the equation of radiation reaction with the running charge $e_{\text{High Field}}$ and the running mass $m_{\text{High Field}}$ under high-intensity fields and also demonstrate the relation $\Xi = \Theta = q(\chi)$ as a plausible candidate. For the realization of QED-based synchrotron radiation, we borrow the result from QED, Eq. (6). At first we consider modification of the LAD field for adopting the QED synchrotron radiation. The equation of an electron's motion and the Maxwell equation with $e_{\text{High Field}}$ and $m_{\text{High Field}}$ become:

$$m_{\text{High Field}}(\tau) \frac{dw^\mu}{d\tau} = -e_{\text{High Field}}(\tau) \mathfrak{F}_{\text{hom}}{}^{\mu\nu} w_\nu \tag{8}$$

$$\partial_\mu F^{\mu\nu} = -c\mu_0 \int_{-\infty}^{\infty} d\tau' e_{\text{High Field}}(\tau') w^\nu(\tau') \delta^4(x' - x(\tau')). \tag{9}$$

Here, $\mathfrak{F}_{\text{hom}} \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4$ is the homogeneous solution of Eq. (9). The solutions of Eq. (9) are the retarded and advanced fields [7,15]:[4]

$$F_{\text{ret}}{}^{\mu\nu}\big|_{x=x(\tau)} = \frac{3}{4} \frac{m_0\tau_0}{ec^2} \left[\frac{d(\Xi w^\mu)}{d\tau} w^\nu - w^\mu \frac{d(\Xi w^\nu)}{d\tau} \right] \int_{-\infty}^{\infty} d\delta\tau \frac{\delta(\delta\tau)}{|\delta\tau|}$$
$$- \frac{m_0\tau_0}{ec^2} \left[\frac{d^2(\Xi w^\mu)}{d\tau^2} w^\nu - \frac{d^2(\Xi w^\nu)}{d\tau^2} w^\mu \right] \tag{10}$$

$$F_{\text{adv}}{}^{\mu\nu}\big|_{x=x(\tau)} = \frac{3}{4} \frac{m_0\tau_0}{ec^2} \left[\frac{d(\Xi w^\mu)}{d\tau} w^\nu - w^\mu \frac{d(\Xi w^\nu)}{d\tau} \right] \int_{-\infty}^{\infty} d\delta\tau \frac{\delta(\delta\tau)}{|\delta\tau|}$$
$$+ \frac{m_0\tau_0}{ec^2} \left[\frac{d^2(\Xi w^\mu)}{d\tau^2} w^\nu - \frac{d^2(\Xi w^\nu)}{d\tau^2} w^\mu \right]. \tag{11}$$

Following Dirac's ideas, "the radiated field $= (F_{\text{ret}}(x) - F_{\text{adv}}(x))/2$" [7], we can obtain the modified LAD field:

$$F_{\text{Mod-LAD}}{}^{\mu\nu}\big|_{x=x(\tau)} = -\frac{m_0\tau_0}{ec^2} \left[\frac{d^2(\Xi w^\mu)}{d\tau^2} w^\nu - \frac{d^2(\Xi w^\nu)}{d\tau^2} w^\mu \right]$$
$$= \Xi \times F_{\text{LAD}}{}^{\mu\nu}\big|_{x=x(\tau)} - \frac{2m_0\tau_0}{ec^2} \frac{d\Xi}{d\tau} \left(\frac{dw^\mu}{d\tau} w^\nu - \frac{dw^\nu}{d\tau} w^\mu \right). \tag{12}$$

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[4] The derivation of this field is based on Ref. [15]. By using the Green function $G_{\text{ret,adv}}(x, x')$, the solution of Maxwell's equation (9) is $A_{\text{ret,adv}}{}^\nu(x) = -ec\mu_0 \int_{-\infty}^{\infty} d\tau' \Xi(\tau') w^\nu(\tau') G_{\text{ret,adv}}(x, x(\tau'))$. The field equations (10)–(11) are derived from the relation $F_{\text{ret,adv}}{}^{\mu\nu}(x) = -ec\mu_0 \int_{-\infty}^{\infty} d\tau' \Xi(\tau') [w^\nu(\tau') \partial^\mu - w^\mu(\tau') \partial^\nu] G_{\text{ret,adv}}(x, x(\tau'))$ at the point $x = x(\tau)$.

We find that this field avoids the singularity of $\int_{-\infty}^{\infty} d\delta\tau \delta(\delta\tau)/|\delta\tau|$. When the factor $\Xi \to 1$, the field $F_{\text{Mod-LAD}} \to F_{\text{LAD}}$ smoothly. Defining the homogeneous field $\mathfrak{F}_{\text{hom}} = F_{\text{ex}} + F_{\text{Mod-LAD}} \in \mathbb{V}_{\text{M}}^4 \otimes \mathbb{V}_{\text{M}}^4$, the equation of an electron's motion, Eq. (8), becomes as follows:

$$m_0 \frac{dw^\mu}{d\tau} = -e \frac{\Xi}{\Theta} \frac{1}{1 - 2\tau_0 \frac{\Xi}{\Theta} \frac{d\Xi}{d\tau}} \mathfrak{F}_{(\Xi)}{}^{\mu\nu} w_\nu. \tag{13}$$

Here $\mathfrak{F}_{(\Xi)} = F_{\text{ex}} + \Xi \times F_{\text{LAD}} \in \mathbb{V}_{\text{M}}^4 \otimes \mathbb{V}_{\text{M}}^4$. Next, we proceed to the demonstration of the relation $\Xi = \Theta = q(\chi)$. At first, we assume that this equation includes the terms of the QED–Sokolov equation. Assuming that the variation of Ξ is very slow, the orders of magnitude of Ξ and Θ are the same, then we obtain $\Xi/\Theta \times |\tau_0 d\Xi/d\tau| \ll 1$. Equation (13) cannot be solved by the same reason as run-away on the LAD equation due to the term of the second-order derivative, the so-called Schott term[5] $m_0\tau_0/(1 - 2\Xi\Theta \times \tau_0 d\Xi/d\tau) \times d^2w/d\tau^2 \in \mathbb{V}_{\text{M}}^4$. For estimating Ξ and Θ, we use perturbation as the method by Landau–Lifshitz [13] with the definition $p = m_0 w \in \mathbb{V}_{\text{M}}^4$:

$$\frac{dp^\mu}{d\tau} = -\frac{e}{m_0} F_{\text{ex}}{}^{\mu\nu} \left(\frac{\Xi}{\Theta} p_\nu + \tau_0 \frac{\Xi^2}{\Theta} g_{\nu\lambda} f_{\text{ex}}{}^\lambda \right) + \frac{\tau_0}{m_0^2 c^2} \frac{\Xi^2}{\Theta} g_{\theta\lambda} f_{\text{ex}}{}^\theta f_{\text{ex}}{}^\lambda p^\mu$$

$$- \frac{e}{m_0} \tau_0 \frac{\Xi^2}{\Theta} \frac{dF_{\text{ex}}{}^{\mu\theta}}{d\tau} p_\theta - 2 \frac{e}{m_0} \tau_0 \frac{\Xi^2}{\Theta^2} \frac{d\Xi}{d\tau} F_{\text{ex}}{}^{\mu\nu} p_\nu + O\left(\tau_0^2\right), \tag{14}$$

where we can find Eq. (7), $\tau_0|_{\text{High Field}} = \Xi^2/\Theta \times \tau_0 = q(\chi) \times \tau_0$, and the direct radiation term $\tau_0 q(\chi)/m_0 \times g_{\alpha\beta} f_{\text{ex}}{}^\alpha f_{\text{ex}}{}^\beta p^\mu$.[6] This direct radiation term also appears in the QED–Sokolov equation (4). For fitting the QED–Sokolov model as mentioned at the beginning of this section,

$$\boxed{\Xi = \Theta} \tag{15}$$

is required since $-e/m_0 \times F_{\text{ex}}{}^{\mu\nu} \left[\Xi/\Theta \times p_\nu + \tau_0 q(\chi) g_{\nu\lambda} f_{\text{ex}}{}^\lambda \right] = -e/m_0 F_{\text{ex}}{}^{\mu\nu} \left[p_\nu + \tau_0 q(\chi) g_{\nu\lambda} f_{\text{ex}}^\lambda \right]$ should be satisfied. The final term on the LHS of Eq. (14) vanishes since $\Xi^2/\Theta^2 \times |\tau_0 d\Xi/d\tau| = |\tau_0 dq(\chi)/d\tau| \ll 1$; the difference between Eq. (14) and the QED–Sokolov model is just $-e/m_0 \times \left[\tau_0 q(\chi) dF_{\text{ex}}{}^{\mu\theta}/d\tau \right] p_\theta$. However, we know that this term (the approximation of the Schott term in the LAD equation) also vanishes in many numerical tests. Therefore, the relation $\Xi = \Theta = q(\chi)$ should be satisfied for describing QED-based synchrotron radiation in the equation of motion. Inserting these relations, Eq. (13) becomes

$$m_0 \frac{dw^\mu}{d\tau} = -e \frac{1}{1 - 2\tau_0 \frac{d\Xi}{d\tau}} \mathfrak{F}_{(\Xi)}{}^{\mu\nu} w_\nu.^{[7]} \tag{16}$$

Equation (16) is one of the methods for the radiation reaction with QED synchrotron radiation; however, it suffers from the run-away problem. I also present the method of stabilizing the singularity of the field $\mathfrak{F} = F_{\text{ex}} + F_{\text{Mod-LAD}}$ before considering the equation of motion in the next section.

2.2. Stabilization by QED vacuum fluctuation

In Sect. 2.1, I modified the LAD field by introducing the running charge $e_{\text{High Field}} = e \times \Xi$, to obtain the modified LAD field $F_{\text{Mod-LAD}} \in \mathbb{V}_{\text{M}}^4 \otimes \mathbb{V}_{\text{M}}^4$. In the following section, we consider how

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[5] The Schott term in the LAD equation is $m_0\tau_0 d^2w/d\tau^2 \in \mathbb{V}_{\text{M}}^4$.

[6] The direct radiation term in the LAD equation is $m_0\tau_0 g_{\alpha\beta} \left(dw^\alpha/d\tau \right)\left(dw^\beta/d\tau \right) w \in \mathbb{V}_{\text{M}}^4$.

[7] Equation (16) derives $dp^\mu/d\tau = -e/m_0 \times F_{\text{ex}}{}^{\mu\nu} \left[p_\nu + \tau_0 q(\chi) g_{\nu\lambda} f_{\text{ex}}{}^\lambda \right] + \tau_0 q(\chi)/m_0^2 c^2 \times g_{\theta\lambda} f_{\text{ex}}{}^\theta f_{\text{ex}}{}^\lambda p^\mu - e/m_0 \times \tau_0 q(\chi) dF_{\text{ex}}{}^{\mu\theta}/d\tau p_\theta - 2e/m_0 \times \tau_0 dq(\chi)/d\tau F_{\text{ex}}{}^{\mu\nu} p_\nu + O(\tau_0^2)$, the quasi-QED–Sokolov equation.

to stabilize the field $\mathfrak{F}_{\text{hom}} = F_{\text{ex}} + F_{\text{Mod-LAD}} \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4 \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4$, which is the homogeneous solution of the source-free Maxwell's equation (9). At first, the field $F_{\text{Mod-LAD}}$ satisfies the source-free Maxwell's equation $\partial_\mu F_{\text{Mod-LAD}}^{\mu\nu} = 0$.[8] Replacing F_{LAD} by $F_{\text{Mod-LAD}}$ using the method of Ref. [10], we can find the homogeneous field $\mathfrak{F}_{\text{hom}} = F_{\text{ex}} + F_{\text{Mod-LAD}}$[9] at an observation point far from an electron. The field dresses the vacuum polarization during the field propagation; $\mathfrak{F}_{\text{hom}}$ represents the already "dressed" field [8–10]. Here we need to derive the undressed field $\mathfrak{F} \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4$ acting on an electron for substitution into Eq. (8). The general dynamics of the propagating field is described by

$$L\left(\langle\mathfrak{F}|\mathfrak{F}\rangle, \langle\mathfrak{F}|^*\mathfrak{F}\rangle\right) = -\frac{1}{4\mu_0}\langle\mathfrak{F}|\mathfrak{F}\rangle + L_{\text{Quantum Vacuum}}\left(\langle\mathfrak{F}|\mathfrak{F}\rangle, \langle\mathfrak{F}|^*\mathfrak{F}\rangle\right). \tag{17}$$

Here, $L_{\text{Quantum Vacuum}}$ is an undefined Lagrangian density for the QED vacuum fluctuation. The important remark is that this Lagrangian density is applicable only to describing the field propagation in the spacetime without any field sources. By solving this, we can obtain the following Maxwell's equation:

$$\partial_\mu\left[\mathfrak{F}^{\mu\nu} + \frac{1}{c\varepsilon_0}M^{\mu\nu}\right] = 0. \tag{18}$$

Equation (18) is the Maxwell's equation for the source-free field, $\mathfrak{F} + M/c\varepsilon_0$, $M \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4$ being the polarization of the vacuum [9,10]:

$$\frac{1}{c\varepsilon_0}M^{\mu\nu} = -\eta f \times \mathfrak{F}^{\mu\nu} - \eta g \times {}^*\mathfrak{F}^{\mu\nu} \tag{19}$$

$$\eta f\left(\langle\mathfrak{F}|\mathfrak{F}\rangle \langle\mathfrak{F}|^*\mathfrak{F}\rangle\right) = 4\mu_0\frac{\partial L_{\text{Quantum Vacuum}}}{\partial\langle\mathfrak{F}|\mathfrak{F}\rangle} \tag{20}$$

$$\eta g\left(\langle\mathfrak{F}|\mathfrak{F}\rangle, \langle\mathfrak{F}|^*\mathfrak{F}\rangle\right) = 4\mu_0\frac{\partial L_{\text{Quantum Vacuum}}}{\partial\langle\mathfrak{F}|\mathfrak{F}\rangle}. \tag{21}$$

Here, $\eta = 4\alpha^2\hbar^3\varepsilon_0/45m_0{}^4c^3$. $\mathfrak{F} + M/c\varepsilon_0$ refers to the dressed-field set of (**D**, **H**). In addition, the following Maxwell's equation also holds: $\partial_\mu\mathfrak{F}_{\text{hom}}{}^{\mu\nu} = 0$. Thus, the solution of Eq. (18), $\mathfrak{F} + M/c\varepsilon_0$, connects to (**D**, **H**) $= \mathfrak{F}_{\text{hom}} = F_{\text{ex}} + F_{\text{Mod-LAD}}$ with continuity and smoothness with C^∞ at all points in the Minkowski spacetime:

$$\boxed{\mathfrak{F}^{\mu\nu} - \eta f \times \mathfrak{F}^{\mu\nu} - \eta g \times {}^*\mathfrak{F}^{\mu\nu} = \mathfrak{F}_{\text{hom}}{}^{\mu\nu}}. \tag{22}$$

Via algebraic treatment, we can solve Eq. (22) for $\mathfrak{F}$:

$$\mathfrak{F}^{\mu\nu} = \frac{(1 - \eta f)\mathfrak{F}_{\text{hom}}{}^{\mu\nu} + \eta g\, ({}^*\mathfrak{F}_{\text{hom}})^{\mu\nu}}{(1 - \eta f)^2 + (\eta g)^2} \tag{23}$$

(see Ref. [10]). We propose the following equation of motion for a radiating electron in high-intensity fields coupling with Maxwell's equation (18):

$$m_{\text{High Field}}(\tau)\frac{dw^\mu}{d\tau} = -e_{\text{High Field}}(\tau)\mathfrak{F}^{\mu\nu}w_\nu. \tag{24}$$

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[8] From the Maxwell's equation $\partial_\mu F_{\text{ret, adv}}^{\mu\nu} = \mu_0\left[-ec\int_{-\infty}^{\infty}d\tau'\,\Xi(\tau)w^\nu\left(\tau'\right)\delta^4\left(x' - x\left(\tau'\right)\right)\right]$, we can define the field $F_{\text{Mod-LAD}}{}^{\mu\nu} = 1/2\left(F_{\text{ret}}{}^{\mu\nu} - F_{\text{adv}}{}^{\mu\nu}\right)\big/2$. See Eqs. (10)–(12).

[9] When $\partial_\mu F_{\text{ex}}{}^{\mu\nu} = 0$, the following are satisfied: $\langle F_{\text{ex}}|F_{\text{ex}}\rangle = 0$ and $\langle F_{\text{ex}}|^*F_{\text{ex}}\rangle = 0$.

Rewriting Eq. (24) with the relation $\Xi = \Theta$ and following Sect. 2.1, we can get the form

$$\boxed{\frac{dw^\mu}{d\tau} = -\frac{e}{m_0} \frac{\left[(1 - \eta f_0) \mathfrak{F}_{\mathrm{hom}}{}^{\mu\nu} + \eta g_0 ({}^*\mathfrak{F}_{\mathrm{hom}})^{\mu\nu} \right]}{(1 - \eta f_0)^2 + (\eta g_0)^2} w_\nu}$$ (25)

(see Appendix A). Here, $f_0 = f\left(\langle \mathfrak{F}_{\mathrm{hom}} | \mathfrak{F}_{\mathrm{hom}} \rangle, \langle \mathfrak{F}_{\mathrm{hom}} |^* \mathfrak{F}_{\mathrm{hom}} \rangle \right)$ and $g_0 = g\left(\langle \mathfrak{F}_{\mathrm{hom}} | \mathfrak{F}_{\mathrm{hom}} \rangle, \langle \mathfrak{F}_{\mathrm{hom}} |^* \mathfrak{F}_{\mathrm{hom}} \rangle \right)$. Introducing the new tensor

$$\mathfrak{K}^{\mu\nu\alpha\beta} = \frac{(1 - \eta f_0) \times g^{\mu\alpha} g^{\nu\beta} + \eta g_0 \times \frac{1}{2!} \varepsilon^{\mu\nu\alpha\beta}}{(1 - \eta f_0)^2 + (\eta g_0)^2},$$ (26)

the equation of an electron's motion briefly becomes

$$\boxed{\frac{dw^\mu}{d\tau} = -\frac{e}{m_0} \mathfrak{K}^{\mu\nu}{}_{\alpha\beta} \mathfrak{F}_{\mathrm{hom}}{}^{\alpha\beta} w_\nu}.$$ (27)

To mimic Sokolov's model, the function Ξ should have the dependence $\Xi \to 1$ in the low-intensity limit converging to Eq. (1).

3. High-intensity field correction under the first-order Heisenberg–Euler vacuum

In this section, we consider Eqs. (25) or (27) with the Heisenberg–Euler model for the QED vacuum fluctuation. After derivation of the equation of an electron's motion, we demonstrate the stability of this equation and perform numerical calculations on it. We choose the relation $\Theta = \Xi = q(\chi)$ in this section; however, we describe it by using Ξ for the extension in Sects. 3.1–3.2.

3.1. Equation of motion

The familiar model of the QED vacuum was represented by Heisenberg and Euler [11,12]:

$$L_{\mathrm{Quantum\ Vacuum}} = L_{\substack{\mathrm{the\ lowest\ order\ of} \\ \mathrm{Heisenberg-Euler}}} = \frac{\alpha^2 \hbar^3 \varepsilon_0{}^2}{360 m_0{}^4 c} \left[4 \langle \mathfrak{F} | \mathfrak{F} \rangle^2 + 7 \langle \mathfrak{F} |^* \mathfrak{F} \rangle^2 \right].$$ (28)

The Heisenberg–Euler Lagrangian basically presents the dynamics only for the constant field. If a more general Lagrangian for any fields exists, that generalized Lagrangian includes a component of Eq. (28), since the constant field should be one of the behaviors of the general Lagrangian. In this section, I assume that we can apply Eq. (28) for the field propagation as in Ref. [10]. In this case, the functions f_0 and g_0 are

$$f_0 = \langle \mathfrak{F}_{\mathrm{hom}} | \mathfrak{F}_{\mathrm{hom}} \rangle = \langle F_{\mathrm{Mod\text{-}LAD}} | F_{\mathrm{Mod\text{-}LAD}} \rangle + 2 \times \langle F_{\mathrm{Mod\text{-}LAD}} | F_{\mathrm{ex}} \rangle$$ (29)

$$g_0 = \frac{7}{4} \langle \mathfrak{F}_{\mathrm{hom}} |^* \mathfrak{F}_{\mathrm{hom}} \rangle = \frac{7}{2} \langle F_{\mathrm{Mod\text{-}LAD}} |^* F_{\mathrm{ex}} \rangle$$ (30)

and we transform Eq. (25) like:

$$\frac{dw^\mu}{d\tau} = -\frac{e}{m_0} \frac{(1 - \eta \langle \mathfrak{F}_{\mathrm{hom}} | \mathfrak{F}_{\mathrm{hom}} \rangle) \mathfrak{F}_{\mathrm{hom}}{}^{\mu\nu} + \frac{7}{4} \eta \langle \mathfrak{F}_{\mathrm{hom}} |^* \mathfrak{F}_{\mathrm{hom}} \rangle ({}^*\mathfrak{F}_{\mathrm{hom}})^{\mu\nu}}{(1 - \eta \langle \mathfrak{F}_{\mathrm{hom}} | \mathfrak{F}_{\mathrm{hom}} \rangle)^2 + \left(\frac{7}{4} \eta \langle \mathfrak{F}_{\mathrm{hom}} |^* \mathfrak{F}_{\mathrm{hom}} \rangle \right)^2} w_\nu.$$ (31)

We can find the singularity when $\eta g_0 = 0$ and $1 - \eta f_0 = 1 - \eta\langle\mathfrak{F}_{\text{hom}}|\mathfrak{F}_{\text{hom}}\rangle = 0$ in Eq. (31). From the condition of the low-intensity limit, $1 - \eta f_0 > 0$ must be required for avoiding the singular point:

$$1 - \eta f_0 = \frac{2\eta}{e^2 c^2} \times \left[m_0 \tau_0 \left(\Xi\ddot{\mathbf{v}} + 2\dot{\Xi}\dot{\mathbf{v}}\right) - e\mathbf{E}_{\text{ex}}\right]^2 \Big|_{\text{rest}} + 1 - \frac{2\eta\mathbf{E}_{\text{ex}}^2}{c^2}\Big|_{\text{rest}}$$

$$> 1 - \frac{2\eta\mathbf{E}_{\text{ex}}^2}{c^2}\Big|_{\text{rest}} \overset{\substack{\text{physical}\\\text{requirements}}}{>} 0, [10] \tag{32}$$

where I have employed $\langle F_{\text{Mod-LAD}}|F_{\text{Mod-LAD}}\rangle = -2\left(m_0\tau_0/ec\right)^2 \times \left(\Xi\ddot{\mathbf{v}} + 2\dot{\Xi}\dot{\mathbf{v}}\right)^2|_{\text{rest}} \leq 0$ and $\langle F_{\text{Mod-LAD}}|F_{\text{ex}}\rangle = 2m_0\tau_0/ec^2 \times \left(\Xi\ddot{\mathbf{v}} + 2\dot{\Xi}\dot{\mathbf{v}}\right)\cdot\mathbf{E}_{\text{ex}}|_{\text{rest}}$. $2\eta\mathbf{E}_{\text{ex}}^2/c^2\big|_{\text{rest}} = \left(5.2\times10^{-5}\right)\times$ $\left(\mathbf{E}_{\text{ex}}\big/E_{\text{Schwinger}}\right)^2\Big|_{\text{rest}}$ with the definition of the Schwinger limit field $E_{\text{Schwinger}} = m_0^2 c^3/\hbar$. Normally, the relation $|\mathbf{E}_{\text{ex}}|_{\text{rest}} << E_{\text{Schwinger}}$ is satisfied. Considering an extreme condition like $|\mathbf{E}_{\text{ex}}|_{\text{rest}} = O\left(E_{\text{Schwinger}}\right)$, the coefficient of $\left(\mathbf{E}_{\text{ex}}/E_{\text{Schwinger}}\right)^2\Big|_{\text{rest}}$ being smaller than unity, $1 - \eta f_0 > 0$, should hold; this is a requirement for avoiding the instability and for taking into consideration the high-intensity fields.

3.2. Run-away avoidance

In the original radiation reaction model, the LAD equation has an instability called the "run-away" solution, diverging exponentially even in the absence of an external field. This mathematical problem is also called "self-acceleration". A new equation is required to hold the stability and we need to understand the size of the dynamical range to which we can apply it. We assume the condition of Eq. (32) in the following analysis. For instance, I rewrite the equation of an electron's motion, Eqs. (25) and (31), as

$$m_0 \frac{dw^\mu}{d\tau} = \frac{(1 - \eta f_0) \times \left[f_{\text{ex}}{}^\mu - eF_{\text{Mod-LAD}}{}^{\mu\nu} w_\nu\right] + \eta g_0 ({}^*f_{\text{ex}})^\mu}{(1 - \eta f_0)^2 + (\eta g_0)^2}, \tag{33}$$

with the definition of the forces $f_{\text{ex}}^\mu = -eF_{\text{ex}}{}^{\mu\nu}w_\nu$ and $({}^*f_{\text{ex}})^\mu = -e\left({}^*F_{\text{ex}}\right)^{\mu\nu}w_\nu$. Here, we follow the two-stage analysis used in Ref. [10]. At first, we check the finiteness of the radiation energy due to the possibility that run-away comes from infinite radiated energy. In the second stage, we proceed to the asymptotic analysis proposed by F. Röhrlich for investigation after release from the external field [16].

In the first stage, we obtain the modified Larmor formula $dW/d\tau = -m_0\tau_0 \Xi g_{\alpha\beta}\left(dw^\alpha/d\tau\right)$ $\left(dw^\beta/d\tau\right)$ by the replacement $\tau_0 \mapsto \Xi \times \tau_0$[11] for the estimation of radiation power from Eq. (33):

$$m_0\tau_0\Xi g_{\mu\nu}\frac{dw^\mu}{d\tau}\frac{dw^\nu}{d\tau} = \frac{\tau_0\Xi}{m_0} \frac{\frac{e^2 c^2}{2\eta}(1 - \eta f_0)^2 \eta f_0 + \frac{2e^2 c^2}{7\eta}(1 - \eta f_0)(\eta g_0)^2}{\left[(1 - \eta f_0)^2 + (\eta g_0)^2\right]^2}$$

$$+ \frac{\tau_0\Xi}{m_0} \frac{g_{\mu\nu}\left[(1 - \eta f_0)f_{\text{ex}}{}^\mu + \eta g_0{}^*f_{\text{ex}}{}^\mu\right]\left[(1 - \eta f_0)f_{\text{ex}}{}^\nu + \eta g_0{}^*f_{\text{ex}}{}^\nu\right]}{\left[(1 - \eta f_0)^2 + (\eta g_0)^2\right]^2}. \tag{34}$$

Considering invariances in the rest frame, $f_0 = -2\left(m_0\tau_0/ec\right)^2 \times \left[m_0\tau_0\left(\Xi\ddot{\mathbf{v}} + 2\dot{\Xi}\dot{\mathbf{v}}\right) - e\mathbf{E}_{\text{ex}}\right]^2\Big|_{\text{rest}} + 2\mathbf{E}_{\text{ex}}^2/c^2\big|_{\text{rest}} = O\left(\Xi^2\ddot{\mathbf{v}}_{\text{rest}}^2\right)$ and $g_0 = 7m_0\tau_0\big/ec \times \left(\Xi\ddot{\mathbf{v}} + 2\dot{\Xi}\dot{\mathbf{v}}\right)\cdot\mathbf{B}_{\text{ex}}|_{\text{rest}} =$

---

[10] The subscript "rest" means the values in the rest frame.
[11] Of course, the well known Larmor formula is $dW/d\tau = -m_0\tau_0 g_{\alpha\beta}\left(dw^\alpha/d\tau\right)\left(dw^\beta/d\tau\right)$.

$O\left(\Xi\ddot{\mathbf{v}}_{\text{rest}}\right)$. When the dynamics becomes run-away by infinite energy emission by light, $|\Xi\ddot{\mathbf{v}}_{\text{rest}}| \to \infty$. In the run-away case, $O\left(|g_0|\right) < O\left(|f_0|\right)$ is obviously satisfied. We use this relation under the condition of Eqs. (32) and (34) (the details can be found in Appendix C):

$$\left| m_0\tau_0\Xi g_{\mu\nu}\frac{dw^\mu}{d\tau}\frac{dw^\nu}{d\tau} \right| \overset{\text{run-away}}{<} \frac{\tau_0\Xi}{m_0}\frac{e^2c^2}{2\eta}\frac{|\eta f_0|}{(1-\eta f_0)^2} + \frac{\tau_0\Xi}{m_0}\frac{2e^2c^2}{7\eta}\frac{1}{1-\eta f_0}\frac{(\eta f_0)^2}{(1-\eta f_0)^2}$$

$$+ \frac{\tau_0\Xi}{m_0}\frac{\left|g_{\mu\nu}f_{\text{ex}}{}^\mu f_{\text{ex}}{}^\nu\right|}{(1-\eta f_0)^2} + \frac{2\tau_0\Xi}{m_0}\frac{|\eta f_0|}{1-\eta f_0}\frac{\times\left|g_{\mu\nu}f_{\text{ex}}{}^\mu\left({}^*f_{\text{ex}}\right)^\nu\right|}{(1-\eta f_0)^2}$$

$$+ \frac{\tau_0\Xi}{m_0}\frac{|\eta f_0|}{(1-\eta f_0)^2}\frac{\left|g_{\mu\nu}f_{\text{ex}}{}^\mu\left(*f_{\text{ex}}\right)^\nu\right|}{1-\eta f_0}. \tag{35}$$

The functions $1/|1-x|$, $1/|1-x|^2$, $|x|/|1-x|^2$, and $|x|^2/|1-x|^2$ are finite in the domain $x \in (-\infty, 1)$. When we choose $x = \eta f_0 \leq 2\eta \left.\mathbf{E}_{\text{ex}}{}^2 \middle/ c^2\right|_{\text{rest}} < O(10^{-5})$ below the Schwinger limit, x is included in this domain. In these conditions,

$$\frac{dW}{dt} = -m_0\tau_0\Xi g_{\mu\nu}\frac{dw^\mu}{d\tau}\frac{dw^\nu}{d\tau} \overset{\text{run-away}}{<} \infty. \tag{36}$$

As such, the possibility of run-away due to the infinite energy emission of light was avoided in Eq. (36).

Next, we proceed to the asymptotic analysis. For this analysis, we need to take the pre-acceleration form. The form of Eq. (33) is as follows:

$$m_0\frac{dw^\mu}{d\tau}(\tau) = e^{\int_{\tau_a}^\tau d\tau''\frac{1}{\Xi\tau_0}}\int_\tau^\infty \frac{d\tau'}{\tau_0}\frac{\begin{bmatrix} (1-\eta f_0)\, f_{\text{ex}}{}^\mu + \eta g_0\left({}^*f_{\text{ex}}\right)^\mu \\ + (1-\eta f_0)\dfrac{m_0\tau_0\Xi}{c^2}g_{\alpha\beta}\dfrac{dw^\alpha}{d\tau}\dfrac{dw^\beta}{d\tau}w^\mu \end{bmatrix}(\tau')}{(1-\eta f_0)\,\Xi}$$

$$\times e^{-\int_{\tau_a}^{\tau'}d\tau''\frac{1}{\Xi\tau_0}} \times e^{-2[\ln\Xi(\tau)-\ln\Xi(\tau')]} \times e^{-\int_{\tau'}^\tau d\tau''\frac{\eta f_0}{\Xi\tau_0}} \times e^{\int_{\tau'}^\tau d\tau''\frac{(\eta g_0)^2}{(1-\eta f_0)\Xi\tau_0}} \tag{37}$$

Here, we have employed the parameter $\tau_a < \tau$. Now we consider the acceleration $dw/d\tau$ at the infinite future, $\tau \to \infty$. Following Röhrlich's method, the acceleration converges to zero when the external field vanishes at $\tau \to \infty$. In this limit, the dynamics becomes the classical limit $\Xi \to 1$ due to the absence of the field ($\chi = 0$), $\lim_{\chi\to 0} q(\chi) = 1$. Therefore, we can obtain the limit of Eq. (37),

$$m_0\frac{dw^\mu}{d\tau}(\infty) = f_{\text{ex}}{}^\mu(\infty) + \frac{\eta g_0}{1-\eta f_0} \times \left({}^*f_{\text{ex}}\right)^\mu(\infty) + \frac{m_0\tau_0\Xi}{c^2}g_{\alpha\beta}\frac{dw^\alpha}{d\tau}\frac{dw^\beta}{d\tau}w^\mu(\infty)$$

$$= \frac{m_0\tau_0\Xi}{c^2}g_{\alpha\beta}\frac{dw^\alpha}{d\tau}\frac{dw^\beta}{d\tau}w^\mu(\infty), \tag{38}$$

by using l'Hôpital's rule at $\tau \to \infty$.[12] The finiteness of Eq. (36) is important for the constant velocity, otherwise $dw/d\tau(\infty) = \infty$. After this procedure, we can follow the method of Ref. [10]. Finally, we can get the limit of the acceleration of an electron:

$$\lim_{\tau\to\infty}\frac{dw^\mu}{d\tau} = 0. \tag{39}$$

This is just the requirement for the avoidance of run-away proposed by Röhrlich [16]. My model also satisfies $dw/d\tau(\infty) = 0$ after release from the external field. In the above, we could demonstrate that the new Eq. (33) does not become run-away under the condition of Eq. (32).

---

[12] For any function f, $\lim_{\tau\to\infty}\int_\tau^\infty \frac{d\tau'}{\tau_0}f\left(\tau'\right)\middle/ e^{-\int_{\tau_a}^\tau d\tau''\frac{1}{\Xi\tau_0}} = \lim_{\tau\to\infty}f(\tau)$.

$$\begin{Bmatrix} \mathbf{E}_{\text{laser}} \\ \mathbf{B}_{\text{laser}} \end{Bmatrix} = \begin{Bmatrix} \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{Bmatrix} \times E_0 \times \exp\left[-\frac{y^2 + z^2}{(\Delta x_{\text{laser spot}})^2}\right]$$

$$\times \exp\left[-\frac{(\omega_{\text{laser}}t - k_{\text{laser}}x)^2}{(\omega_{\text{laser}}\Delta\tau)^2}\right] \times \sin(\omega_{\text{laser}}t - k_{\text{laser}}x)$$

Fig. 1. Setup of the laser–electron "head-on collision". The laser photons propagate along the x axis. An electron travels in the negative x direction.

3.3. Calculations

Finally, we present the numerical calculation results with other radiation reaction models. Employing the relation $\Xi = q(\chi)$, Eq. (33) becomes

$$m_0 \frac{dw^\mu}{d\tau} = \frac{(1 - \eta f_0) \times [f_{\text{ex}}{}^\mu + q(\chi) \times f_{\text{LAD}}{}^\mu] + \eta g_0 \, (^*f_{\text{ex}})^\mu}{(1 - \eta f_0)^2 + (\eta g_0)^2 - 2\tau_0 \, (1 - \eta f_0) \frac{dq(\chi)}{d\tau}}, \qquad (40)$$

where $\mathfrak{F}_{q(\chi)} = F_{\text{ex}} + q(\chi)F_{\text{LAD}}$. We have performed the calculation of Eq. (40) with the following models: the Seto I model, which is Eq. (1) [10], the Landau–Lifshitz model [13], classical Sokolov [17], and QED–Sokolov [14]. In addition, we call Eqs. (25), (33), (40) "Seto II". I have assumed the case of a head-on collision between the laser photons and an electron as the initial configuration of the simulations (Fig. 1). We used the parameters of the Extreme Light Infrastructure– Nuclear Physics (ELI–NP) [5,6]. The peak intensity of the laser is 1×10^{22} W/cm^2 in a Gaussian- shaped plane-wave like Eqs. (28), (29) in Ref. [10]. The pulse width is 22 fsec and the laser wavelength is $0.82\,\mu$m. The electric field is situated in the y direction, the magnetic field is in the z direction. The single electron travels in the negative x direction, with an initial energy of 600 MeV. The numerical calculations were carried out in the laboratory frame.

The time evolution of an electron's energy shows the typical behavior of the radiation reaction, as shown in Fig. 2. The energy of an electron drops from the initial energy of 600 MeV. Depending on the models, the final energies of the electron converge to two separate levels. The first group includes the Seto I, Landau–Lifshitz, and classical Sokolov models near 165 MeV. The second group is the QED–Sokolov and Seto II models, starting around 260 MeV. The difference between these two groups obviously depends on the function $q(\chi)$. At this laser intensity and electron energy, χ runs from 0 to 0.3 in this case. Figure 3 presents the graph of $q(\chi)$.

The following is satisfied: $\tau_0 dq(\chi)/d\tau = O(10^{-5})$ (see Fig. 4) and also $\eta f_0 = \eta \langle \mathfrak{F}_{\text{hom}} | \mathfrak{F}_{\text{hom}} \rangle = O(10^{-8})$. Therefore, $1 - \eta f_0 - 2\tau_0 dq(\chi)/d\tau \sim 1$ in this case. In the head-on collision between the laser photon and the electron, $\eta g_0 = 7/4 \times \eta \langle \mathfrak{F}_{\text{hom}} | ^*\mathfrak{F}_{\text{hom}} \rangle = 7m_0\tau_0 q(\chi)/ec \times \ddot{\mathbf{v}} \cdot \mathbf{B}_{\text{ex}}|_{\text{rest}} = 0$ since the initial condition limits the electron's motion on the x–y plane and $\mathbf{B}_{\text{ex}}$ in the z direction. Rounding the invariance ηf_0 into 1, Eq. (40) is reduced as follows:

$$m_0 \frac{dw^\mu}{d\tau} = -e \frac{1}{1 - \eta \langle \mathfrak{F}_{\text{hom}} | \mathfrak{F}_{\text{hom}} \rangle - 2\tau_0 \frac{dq(\chi)}{d\tau}} \mathfrak{F}_{q(\chi)}{}^{\mu\nu} w_\nu \sim -e \frac{1}{1 - 2\tau_0 \frac{dq(\chi)}{d\tau}} \mathfrak{F}_{q(\chi)}{}^{\mu\nu} w_\nu. \quad (41)$$
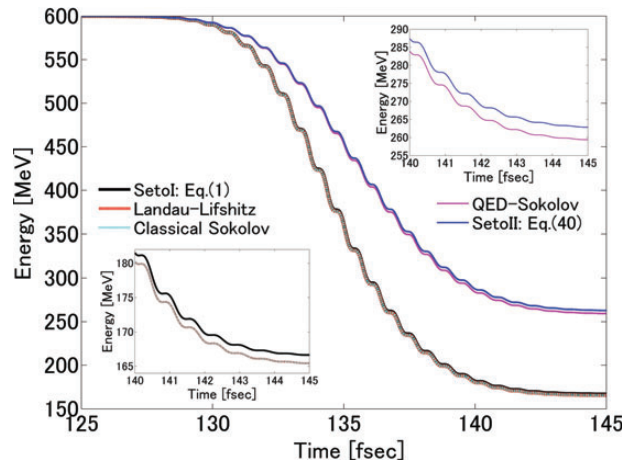
Fig. 2. The time evolution of an electron's energy. The final electron energies are Seto I, Eq. (1): 166.5 MeV; Landau–Lifshitz: 165.3 MeV; classical Sokolov: 165.3 MeV; QED–Sokolov: 259.0 MeV; and Seto II, Eq. (40): 262.5 MeV. The insets are close-ups of the figure.
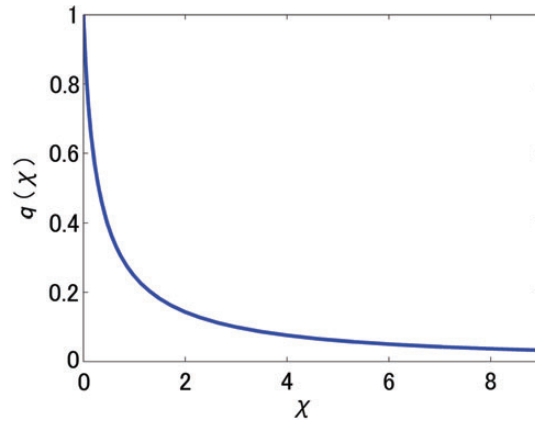

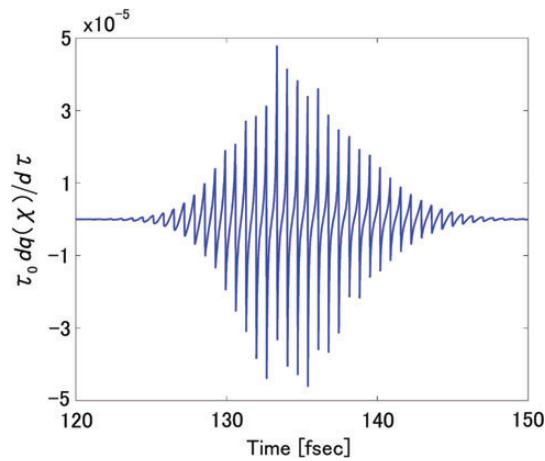**Fig. 3.** The function of $q(\chi)$. In this calculation, the domain is $\chi \in [0, 1]$.

Fig. 4. The function of $\tau_0 dq(\chi)/d\tau$.

Since Eq. (16) is valid, we can derive the quasi-QED–Sokolov equation by using perturbation[13] from Eq. (40). The convergence between the Seto II, Eq. (40), and QED–Sokolov, Eqs. (4)–(6), models appeared for this reason. The difference in the final energies between the two groups depends on the invariant function $\Xi = q(\chi)$, $I_{QED} = q(\chi) \times I_{classic} \leq I_{classic}$.[14] From the theoretical point of view, the new equation (40) can satisfy both the relation $(dx^\mu/d\tau)(dx_\mu/d\tau) = c^2$ and $p_\mu p^\mu = m_0^2 c^2$[15] at any time. On the other hand, in the classical/QED–Sokolov model, $(dx^\mu/d\tau)(dx_\mu/d\tau) \neq c^2$ and $p_\mu p^\mu = m_0^2 c^2$ (see Eq. (4))]. This is an algebraic difference between the two models.

4.   Conclusion and discussion

In summary, I have updated my previous equation of a radiating electron's motion by considering the high-intensity fields and QED vacuum fluctuation. The field $\mathfrak{F}$ acting on an electron was modified by the following method:

$$
\left.\begin{array}{l}
\text{1) } F_{LAD} \mapsto F_{\text{Mod-LAD}} \text{ (Sect. 2.1)} \\
\text{2) QED vacuum (Sect. 2.2)}
\end{array}\right\}
\quad
\begin{array}{c}
\text{3) } \Xi = q(\chi) \\
\Rightarrow \\
\text{(Sect. 3.3/Appendix A)}
\end{array}
\quad
\frac{dw^\mu}{d\tau} = -\frac{e}{m_0} \mathfrak{K}^{\mu\nu}{}_{\alpha\beta} \left(F_{ex}{}^{\alpha\beta} + F_{\text{Mod-LAD}}{}^{\alpha\beta} \right) w_\nu.
$$

The external high-intensity fields modify the emitted field from an electron. The QED vacuum fluctuation stabilizes the "run-away". The mathematical treatment in the derivation of the new equation was based on our previous papers [9,10]. At first, I assumed the parameter replacements $e \mapsto e_{\text{High Field}} = e \times \Xi$ and $m_0 \mapsto m_{\text{High Field}} = m_0 \times \Theta$ in the high-intensity field for taking the QED-intensity correction into the formula. In the low-intensity limit, the invariants satisfy $\Xi, \Theta \to 1$. The source term of Maxwell's equation was deformed, and depending on this replacement, the LAD field was modified from F_{LAD} to $F_{\text{Mod-LAD}}$ (Sect. 2.1). The field $\mathfrak{F}$, which acts on an electron in the QED vacuum fluctuation, should satisfy the following equation:

$$
\boxed{\mathfrak{F}^{\mu\nu} - \eta f \times \mathfrak{F}^{\mu\nu} - \eta g \times {}^*\mathfrak{F}^{\mu\nu} = F_{ex}{}^{\mu\nu} + F_{\text{Mod-LAD}}{}^{\mu\nu}}. \tag{42}
$$

Then we get the following new equation of motion of an electron including the radiation reaction:

$$
\boxed{\frac{dw^\mu}{d\tau} = -\frac{e}{m_0} \mathfrak{K}^{\mu\nu}{}_{\alpha\beta} \mathfrak{F}_{\text{hom}}{}^{\alpha\beta} w_\nu}, \tag{43}
$$

or the explicit form:

$$
\frac{dw^\mu}{d\tau} = -\frac{e}{m_0} \frac{\left[(1 - \eta f_0)\, \mathfrak{F}_{\text{hom}}{}^{\mu\nu} + \eta g_0\, ({}^*\mathfrak{F}_{\text{hom}})^{\mu\nu} \right] w_\nu}{(1 - \eta f_0)^2 + (\eta g_0)^2}. \tag{44}
$$

Here, the definition of the homogeneous field $\mathfrak{F}_{\text{hom}}$ is $\mathfrak{F}_{\text{hom}} = F_{ex} + F_{\text{Mod-LAD}}$ (Sect. 2.2). And, we have employed the relation $\Xi = \Theta = q(\chi)$ from QED-based synchrotron radiation. From the analysis, the following relation must be satisfied for the stability of this equation in the Heisenberg–Euler vacuum [11,12]: $1 - 2\eta\, \mathbf{E}_{ex}^2 / c^2\big|_{\text{rest}} > 0$ (Eq. (32) in Sect. 3.1). Under this condition, we could demonstrate the run-away avoidance by using the Röhrlich method [16] (Sect. 3.2), and we could perform a numerical calculation for checking the difference between the proposed models.

---

[13] "Perturbation" means the replacements $dw/dt \mapsto f_{ex}/m_0$ on the RHS of Eq. (40) [13].

[14] $I_{\text{classical}} = -dW/d\tau|_{\text{classical}} = -m_0\tau_0 g_{\alpha\beta} (dw^\alpha/d\tau)(dw^\beta/d\tau)$.

[15] $p = m_0 w = m_0 dx/d\tau \in \mathbb{V}_M^4$ for my new model.

The results showed the dependence of $q(\chi)$, the correction in high-intensity fields being the essential difference between the models (Sect. 3.3). This equation only requires that the field strength of the external field is smaller than the Schwinger limit. Of course, this equation also maintains the invariance $\left(dx^\mu \big/ dx^\mu \right) \left(dx_\mu \big/ d\tau \right) = c^2$, which the QED–Sokolov equation cannot satisfy. In the results of the numerical simulation and analysis, the proposed Eqs. (43) and (44) can include the dynamics of the QED–Sokolov equation (4)–(6) as long as we choose the relation $\Xi = \Theta = q(\chi)$.

We introduce the measure of an electron's mass $m(x)$ and an anisotropic electron's charge $\mathfrak{E}(x) \in \mathbb{V}_{\mathrm{M}}^4 \otimes \mathbb{V}_{\mathrm{M}}^4 \otimes \mathbb{V}_{\mathrm{M}}^4 \otimes \mathbb{V}_{\mathrm{M}}^4$ following Ref. [10]:

$$dm(x)\frac{dw^\mu}{d\tau} = -d\mathfrak{E}^{\mu\nu}{}_{\alpha\beta}(x)\mathfrak{F}_{\mathrm{hom}}{}^{\alpha\beta}w_\nu. \tag{45}$$

The Radon–Nikodym derivative [18] should be $d\mathfrak{E}^{\mu\nu}{}_{\alpha\beta}\big/dm = e_{\mathrm{High\ Field}}\big/m_{\mathrm{High\ Field}} \times \mathfrak{K}^{\mu\nu}{}_{\alpha\beta}$, depending on the invariance Ξ. The anisotropy of charge $\mathfrak{E}(x)$ comes from the polarization of the QED vacuum. This effect is a unique dynamics that the QED–Sokolov equations (3)–(5) do not include. This Radon–Nikodym derivative $d\mathfrak{E}/dm$ is the "QED vacuum filter of fields" between the dressed and undressed fields [10].

In the Heisenberg–Euler vacuum [11,12] (Sect. 3.1), we can find the limit of the photon energy (as in Ref. [9]). From the limit $1 - 2\eta\mathbf{E}_{\mathrm{ex}}{}^2\big/c^2\big|_{\mathrm{rest}} > 0$, the field strength should satisfy $|\mathbf{E}_{\mathrm{ex}}|_{\mathrm{rest}} < E_{\mathrm{Schwinger}}$ with the definition of the Schwinger limit $E_{\mathrm{Schwinger}} = m_0{}^2c^3\big/e\hbar$. On the other hand, the limit of the photon energy comes from the expansion of the Heisenberg–Euler action integral [9]:

$$\int d^4x\, L_{\mathrm{Heisenberg-Euler}}^{\text{1st order}} \left(\langle \mathfrak{F}|\mathfrak{F} \rangle, \langle \mathfrak{F}|^*\mathfrak{F} \rangle \right)$$

$$= -\int d^4x\, \frac{1}{4\mu_0} \langle \mathfrak{F}|\mathfrak{F} \rangle + \int d^4x\, \frac{\alpha^2\hbar^3\varepsilon_0{}^2}{360 m_0{}^4 c} \left[4\langle \mathfrak{F}|\mathfrak{F} \rangle^2 + 7\langle \mathfrak{F}|^*\mathfrak{F} \rangle^2 \right] + O\left(\left(\frac{g_{\mu\nu}\hbar k^\mu \hbar k^\nu}{m_0{}^2 c^2} \right)^3 \right) \tag{46}$$

For this expansion, $2g_{\mu\nu}\hbar k_{\mathrm{laser}}{}^\mu \hbar k_{\mathrm{radiation}}{}^\nu < 4 \times \hbar\omega_{\mathrm{laser}} \times \hbar\omega_{\mathrm{radiation}}/c^2 < m_0{}^2c^2 = (0.5\,\mathrm{MeV}/c)^2$. In the numerical simulation of the head-on collision, we used a laser wavelength of $0.82\,\mu\mathrm{m}$, equivalent to 1.5 eV; therefore, the maximum radiated photon energy is $\hbar\omega_{\mathrm{radiation}} < O(10\,\mathrm{GeV})$ [9]. These are the phenomenological limits of Maxwell's equation in a QED vacuum, in the proposed model. Exceeding this limit is equivalent to breaking the QED vacuum with electron–positron pair creation by energetic photons.

Finally, we discuss the experiments for checking radiation reaction models. In this paper, we have used the plausible relation $\Xi = q(\chi)$, which converges the dynamics to Sokolov's equation [14]. Under $\tau_0 d\Xi/d\tau \ll 1$ for a simplification, we can observe the radiated field from an electron:

$$F_{\mathrm{Mod\text{-}LAD}}{}^{\mu\nu} \sim \Xi \times F_{\mathrm{LAD}}{}^{\mu\nu}. \tag{47}$$

Ξ can round the QED effects into the classical dynamics via Eqs. (8)–(9) by the present proposal. We may find the relation $F_{\mathrm{Mod\text{-}LAD}}{}^{\mu\nu} \sim [q(\chi) + \delta q] \times F_{\mathrm{LAD}}{}^{\mu\nu}$, where δq denotes the alteration from the synchronization. In this case, we shall go back to the equation

$$\frac{dw^\mu}{d\tau} = -\frac{e_{\mathrm{High\ Field}}}{m_{\mathrm{High\ Field}}} \frac{(1 - \eta f_0)\,\mathfrak{F}_{\mathrm{hom}}{}^{\mu\nu}w_\nu + \eta g_0\,(^*\mathfrak{F}_{\mathrm{hom}})^{\mu\nu}w_\nu}{(1 - \eta f_0)^2 + (\eta g_0)^2}$$

$$= -\frac{e}{m_0}\frac{\Xi}{\Theta}\frac{(1 - \eta f_0)\,\mathfrak{F}_{\mathrm{hom}}{}^{\mu\nu}w_\nu + \eta g_0\,(^*\mathfrak{F}_{\mathrm{hom}})^{\mu\nu}w_\nu}{(1 - \eta f_0)^2 + (\eta g_0)^2}, \tag{48}$$

before putting the relation $\Xi = \Theta$. The relation Eq. (47) is replaced by $F_{\text{Mod-LAD}}{}^{\mu\nu} \sim \Xi/\Theta \times F_{\text{LAD}}{}^{\mu\nu} = [q(\chi) + \delta q] \times F_{\text{LAD}}{}^{\mu\nu}$. Since QED-based synchrotron radiation is the general electromagnetic interaction between an electron and the external fields, we can assume $\Xi = q(\chi)$ from the main discussion in this paper. Hence, $\Theta = \Xi/[q(\chi) + \delta q] \sim 1 - \delta q/q(\chi)$ is estimated. For example, one of the candidates for this effect is non-electromagnetic interactions like the Poincaré stress, which is introduced by the inner structure of an electron for the stabilization of its electromagnetic mass [19], so we can also extend this model to unknown non-electromagnetic interactions. Since $d\mathfrak{E}^{\mu\nu}{}_{\alpha\beta}/dm = e_{\text{High Field}}/m_{\text{High Field}} \times \mathfrak{K}^{\mu\nu}{}_{\alpha\beta} = e/m_0 \times \Xi/\Theta \times \mathfrak{K}^{\mu\nu}{}_{\alpha\beta}$ represents a QED coupling correction between an electron and radiation in high-intensity fields, the investigation of Ξ and Θ will become more important for the radiation reaction acting on an electron in ultra-high-intensity fields.

Acknowledgements

Appendix A.   Details of the derivation of the equation of motion, Eq. (25)

Here we derive the undressed field $\mathfrak{F} \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4$ from Eq. (23):

$$\mathfrak{F}^{\mu\nu} = \frac{(1 - \eta f)\,\mathfrak{F}_{\text{hom}}{}^{\mu\nu} + \eta g\,({}^*\mathfrak{F}_{\text{hom}})^{\mu\nu}}{(1 - \eta f)^2 + (\eta g)^2}. \tag{A1}$$

The definitions of the invariant function f and g are Eqs. (20) and (21). We can expect the form of

$$\mathfrak{F}^{\mu\nu} = \mathfrak{F}_{\text{hom}}{}^{\mu\nu} + (\delta \times \theta)\mathfrak{F}_{\text{hom}}{}^{\mu\nu} + (\delta \times \theta_*)\left({}^*\mathfrak{F}_{\text{hom}}\right)^{\mu\nu}. \tag{A2}$$

We assume that the parameter δ satisfies the relation $|\delta| \ll 1$. The functions θ and θ_* depend on $\mathfrak{F}$. This relation leads to the expansion of the invariant functions:

$$f\left(\langle\mathfrak{F}|\mathfrak{F}\rangle, \langle\mathfrak{F}|{}^*\mathfrak{F}\rangle\right) = f\left(\langle\mathfrak{F}_{\text{hom}}|\mathfrak{F}_{\text{hom}}\rangle, \langle\mathfrak{F}_{\text{hom}}|{}^*\mathfrak{F}_{\text{hom}}\rangle\right) + \delta \times \Theta_f \tag{A3}$$

$$g\left(\langle\mathfrak{F}|\mathfrak{F}\rangle, \langle\mathfrak{F}|{}^*\mathfrak{F}\rangle\right) = g\left(\langle\mathfrak{F}_{\text{hom}}|\mathfrak{F}_{\text{hom}}\rangle, \langle\mathfrak{F}_{\text{hom}}|{}^*\mathfrak{F}_{\text{hom}}\rangle\right) + \delta \times \Theta_g. \tag{A4}$$

For instance, we introduce $f_0 = f\left(\langle\mathfrak{F}_{\text{hom}}|\mathfrak{F}_{\text{hom}}\rangle, \langle\mathfrak{F}_{\text{hom}}|{}^*\mathfrak{F}_{\text{hom}}\rangle\right)$ and $g_0 = g\left(\langle\mathfrak{F}_{\text{hom}}|\mathfrak{F}_{\text{hom}}\rangle, \langle\mathfrak{F}_{\text{hom}}|{}^*\mathfrak{F}_{\text{hom}}\rangle\right)$. By using these equations, Eq. (A1) becomes

$$\begin{aligned}
\mathfrak{F}^{\mu\nu} &= \frac{\left(1 - \eta f_0 - \eta\delta \times \Theta_f\right)\mathfrak{F}_{\text{hom}}{}^{\mu\nu} + \left(\eta g_0 + \eta\delta \times \Theta_g\right)({}^*\mathfrak{F}_{\text{hom}})^{\mu\nu}}{(1 - \eta f_0)^2 + (\eta g_0)^2} \\
&\quad \times \sum_{n=0}^{\infty} \eta^n \delta^n \left[\frac{2\Theta_f - 2\eta\left(f_0\Theta_f + g_0\Theta_g\right) - \eta\delta\left(\Theta_f{}^2 + \Theta_g{}^2\right)}{(1 - \eta f_0)^2 + (\eta g_0)^2}\right]^n \\
&= \frac{(1 - \eta f_0)\,\mathfrak{F}_{\text{hom}} + \eta g_0\,({}^*\mathfrak{F}_{\text{hom}})^{\mu\nu} + O(\eta\delta)}{(1 - \eta f_0)^2 + (\eta g_0)^2}.
\end{aligned} \tag{A5}$$

By neglecting the terms of $O(\eta\delta)$,[16]

$$\mathfrak{F}^{\mu\nu} = \frac{(1-\eta f_0)\,\mathfrak{F}_{\text{hom}} + \eta g_0\,(^*\mathfrak{F}_{\text{hom}})^{\mu\nu}}{(1-\eta f_0)^2 + (\eta g_0)^2}, \tag{A6}$$

with the definition as follows:

$$\mathfrak{F}_{\text{hom}} = F_{\text{ex}} + F_{\text{Mod-LAD}}, \tag{A7}$$

$$F_{\text{Mod-LAD}}^{\mu\nu} = -\frac{m_0\tau_0}{ec^2}\left[\frac{d^2\,(\Xi w^\mu)}{d\tau^2}w^\nu - \frac{d^2\,(\Xi w^\nu)}{d\tau^2}w^\mu\right]. \tag{A8}$$

By substituting $\mathfrak{F}$ into the equation of motion,

$$m_{\text{High Field}}(\tau)\frac{dw^\mu}{d\tau} = -e_{\text{High Field}}(\tau)\mathfrak{F}^{\mu\nu}w_\nu \tag{A9}$$

becomes

$$\frac{dw^\mu}{d\tau} = -\frac{e_{\text{High Field}}(\tau)}{m_{\text{High Field}}(\tau)}\frac{(1-\eta f_0)\,\mathfrak{F}_{\text{hom}}{}^{\mu\nu}w_\nu + \eta g_0\,(^*\mathfrak{F}_{\text{hom}})^{\mu\nu}w_\nu}{(1-\eta f_0)^2 + (\eta g_0)^2}. \tag{A10}$$

In the case of the low-intensity limit, the charge-to-mass ratio should be $e_{\text{High Field}}/m_{\text{High Field}} = e/m_0 = 1.75 \times 10^{11}$ [C/kg]. How is it in the case of high-intensity fields? By transforming Eq. (A10),

$$m_0\frac{dw^\mu}{d\tau} = \frac{\Xi}{\Theta}\frac{(1-\eta f_0)\times[f_{\text{ex}}{}^\mu + \Xi \times f_{\text{LAD}}{}^\mu] + \eta g_0\,(^* f_{\text{ex}})^\mu}{(1-\eta f_0)^2 + (\eta g_0)^2 - 2\tau_0\,(1-\eta f_0)\frac{\Xi}{\Theta}\frac{d\Xi}{d\tau}}. \tag{A11}$$

Following Sect. 2.1, we choose the relation ηf_0, ηg_0; $\Xi/\Theta \times \tau_0 d\Xi/d\tau$ is sufficiently smaller than unity and Ξ and Θ are of the same order of the magnitude. By this choice we can reduce this equation as

$$m_0\frac{dw^\mu}{d\tau} \approx \frac{\Xi}{\Theta}\times f_{\text{ex}}{}^\mu + \frac{\Xi^2}{\Theta}\times f_{\text{LAD}}{}^\mu = \frac{\Xi}{\Theta}\times f_{\text{ex}}{}^\mu + \frac{\Xi^2}{\Theta}\times m_0\tau_0\frac{d^2w^\mu}{d\tau^2} + \frac{\Xi^2}{\Theta}$$
$$\times \frac{m_0\tau_0}{c^2}g_{\alpha\beta}\frac{dw^\alpha}{d\tau}\frac{dw^\beta}{d\tau}w^\mu. \tag{A12}$$

Therefore, the Larmor radiation formula obtains the correction factor of Ξ^2/Θ:

$$\left.\frac{dW}{dt}\right|_{\text{High Field}} = -\frac{\Xi^2}{\Theta}\times m_0\tau_0 g_{\alpha\beta}\frac{dw^\alpha}{d\tau}\frac{dw^\beta}{d\tau}. \tag{A13}$$

Following the QED-based synchrotron radiation formula,

$$\left.\frac{dW}{dt}\right|_{\text{High Field}} = q(\chi)\times\left.\frac{dW}{dt}\right|_{\text{Classical}}, \tag{A14}$$

$$q(\chi) = \frac{9\sqrt{3}}{8\pi}\int_0^{\chi^{-1}} dr\,r\left[\int_{r_\chi}^\infty dr'\,K_{5/3}\left(r'\right) + \chi^2 r r_\chi K_{2/3}\left(r_\chi\right)\right], \tag{A15}$$

we can find the candidate $\Xi^2/\Theta = q(\chi)$. In Eq. (A12), $\Xi/\Theta \times f_{\text{ex}}$ should be f_{ex} in the QED–Sokolov model, $\Theta = \Xi$. Therefore, we can propose the relation $\Theta = \Xi \Rightarrow e_{\text{High Field}}/m_{\text{High Field}} = e/m_0$ and the equation of a radiating electron's motion,

$$\frac{dw^\mu}{d\tau} = \frac{e}{m_0}\frac{(1-\eta f_0)\,\mathfrak{F}_{\text{hom}}{}^{\mu\nu}w_\nu + \eta g_0\,(^*\mathfrak{F}_{\text{hom}})^{\mu\nu}w_\nu}{(1-\eta f_0)^2 + (\eta g_0)^2}. \tag{A16}$$

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[16] This order cut-off is important for the stability of the new equation. See Sects. 3.1 and 3.2.

Conversely, the choices of $\Theta = \Xi$ and $\Xi = q(\chi)$ satisfy the relations ηf_0, $\eta g_0 \ll 1$, and $\Xi/\Theta \times |\tau_0 d\Xi/d\tau| \ll 1$.

Appendix B.   Errata of K. Seto, Prog. Theor. Exp. Phys. 2015, 023A01 (2015) [10]

Strictly speaking, Eq. (1) should be

$$\frac{dw^\mu}{d\tau} \overset{\Xi\equiv 1}{=} -\frac{e}{m_0} \frac{\left[(1-\eta f_0)\,\mathfrak{F}^{\mu\nu} + \eta g_0\,(^*\mathfrak{F})^{\mu\nu}\right] w_\nu}{(1-\eta f_0)^2 + (\eta g_0)^2}, \tag{B1}$$

for connecting from Eq. (44) or Eq. (A16). Here, $\mathfrak{F} = \mathfrak{F}_{\mathrm{hom}}|_{\Xi\equiv 1} = F_{\mathrm{ex}} + F_{\mathrm{LAD}} \in \mathbb{V}_M{}^4 \otimes \mathbb{V}_M{}^4$. Normally, $|1-\eta f_0| \gg |\eta g_0|$ is satisfied, and Eq. (B1) is transformed as follows under this condition:

$$\frac{dw^\mu}{d\tau} \overset{\Xi\equiv 1}{=} -\frac{e}{m_0(1-\eta f_0)} \frac{\mathfrak{F}^{\mu\nu} + \frac{\eta g_0}{1-\eta f_0}(^*\mathfrak{F})^{\mu\nu}}{1 + \frac{(\eta g_0)^2}{(1-\eta f_0)^2}} w_\nu = -\frac{e}{m_0\,(1-\eta f_0)}\left[\mathfrak{F}^{\mu\nu} + \eta g_0\,(^*\mathfrak{F})^{\mu\nu}\right] w_\nu + \cdots. \tag{B2}$$

This is the Eq. (1) derived in Ref. [10]. However, in the strict order expansion, it should be

$$\frac{dw^\mu}{d\tau} \overset{\Xi\equiv 1}{=} -\frac{e}{m_0\,(1-\eta f_0)}\mathfrak{F}^{\mu\nu}w_\nu + \mathrm{O}\left(\frac{\eta g_0}{1-\eta f_0}\right). \tag{B3}$$

In this form, the analysis of run-away avoidance in Ref. [10] becomes easier,[17] and the numerical calculation almost agrees since $|\mathfrak{F}^{\mu\nu}| \gg \left|\eta g_0\,(^*\mathfrak{F})^{\mu\nu}\right|$ is satisfied. We suggested the anisotropy of the QED vacuum. We can confirm the anisotropic field $^*\mathfrak{F}$ in the form of Eq. (B1), but it does not exist in (B3). The higher orders of $\eta g_0/(1-\eta f_0)$ describe the degree of anisotropy of the QED vacuum.

Appendix C.   The details of Eq. (35)

From Eq. (34),

$$m_0\tau_0 g_{\mu\nu}\frac{dw^\mu}{d\tau}\frac{dw^\nu}{d\tau} = \frac{\tau_0}{m_0}\frac{\frac{e^2c^2}{2\eta}(1-\eta f_0)^2\,\eta f_0 + \frac{2e^2c^2}{7\eta}(1-\eta f_0)\,(\eta g_0)^2}{\left|(1-\eta f_0)^2 + (\eta g_0)^2\right|^2}$$
$$+ \frac{\tau_0}{m_0}\frac{g_{\mu\nu}\left[(1-\eta f_0)\,f_{\mathrm{ex}}^\mu + \eta g_0{}^*f_{\mathrm{ex}}^\mu\right]\left[(1-\eta f_0)\,f_{\mathrm{ex}}^\nu + \eta g_0{}^*f_{\mathrm{ex}}^\nu\right]}{\left|(1-\eta f_0)^2 + (\eta g_0)^2\right|^2}. \tag{C1}$$

We consider this equation under the condition of Eq. (32):

$$1 - \eta f_0 \overset{\substack{\text{physical}\\\text{requirements}}}{>} 0. \tag{C2}$$

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[17] For considering Eq. (B2), we only put the relation $g_0 = 0$ in Sect. 3 in Ref. [10] and any anisotropy vanishes.

This condition supports the relation $(1 - \eta f_0)^2 + (\eta g_0)^2 > 0$. Considering the above, we can proceed to consider the absolute value of Eq. (C1):

$$\left| m_0 \tau_0 g_{\mu\nu} \frac{dw^\mu}{d\tau} \frac{dw^\nu}{d\tau} \right| = \frac{\tau_0}{m_0} \frac{\frac{e^2 c^2}{2\eta} (1 - \eta f_0)^2 |\eta f_0| + \frac{2e^2 c^2}{7\eta} (1 - \eta f_0) (\eta g_0)^2}{\left| (1 - \eta f_0)^2 + (\eta g_0)^2 \right|^2}$$

$$+ \frac{\tau_0}{m_0} \frac{\left| g_{\mu\nu} \left[(1 - \eta f_0) f_{\mathrm{ex}}{}^\mu + \eta g_0 \, (^* f_{\mathrm{ex}})^\mu \right] \left[(1 - \eta f_0) f_{\mathrm{ex}}{}^\nu + \eta g_0 \, (^* f_{\mathrm{ex}})^\nu \right] \right|}{\left| (1 - \eta f_0)^2 + (\eta g_0)^2 \right|^2}$$

$$\leq \frac{\tau_0}{m_0} \frac{\frac{e^2 c^2}{2\eta} (1 - \eta f_0)^2 |\eta f_0| + \frac{2e^2 c^2}{7\eta} (1 - \eta f_0) (\eta g_0)^2}{(1 - \eta f_0)^4}$$

$$+ \frac{\tau_0}{m_0} \frac{\left| g_{\mu\nu} \left[(1 - \eta f_0) f_{\mathrm{ex}}{}^\mu + \eta g_0 \, (^* f_{\mathrm{ex}})^\mu \right] \left[(1 - \eta f_0) f_{\mathrm{ex}}{}^\nu + \eta g_0 \, (^* f_{\mathrm{ex}})^\nu \right] \right|}{(1 - \eta f_0)^4}$$

$$\leq \frac{\tau_0}{m_0} \frac{\frac{e^2 c^2}{2\eta} |\eta f_0|}{(1 - \eta f_0)^2} + \frac{\tau_0}{m_0} \frac{\frac{2e^2 c^2}{7\eta} (\eta g_0)^2}{(1 - \eta f_0)^3} + \frac{\tau_0}{m_0} \frac{|g_{\mu\nu} f_{\mathrm{ex}}{}^\mu f_{\mathrm{ex}}{}^\nu|}{(1 - \eta f_0)^2}$$

$$+ \frac{2\tau_0}{m_0} \frac{|\eta g_0|}{1 - \eta f_0} \frac{\times |g_{\mu\nu} f_{\mathrm{ex}}{}^\mu (^* f_{\mathrm{ex}})^\nu|}{(1 - \eta f_0)^2} + \frac{\tau_0}{m_0} \frac{\eta g_0}{1 - \eta f_0} \frac{\left| g_{\mu\nu} f_{\mathrm{ex}}{}^\mu (^* f_{\mathrm{ex}})^\nu \right|}{(1 - \eta f_0)^2}. \tag{C3}$$

Finally, $O(|g_0|) < O(|f_0|)$ is satisfied in the case of run-away, and we can obtain the relation of Eq. (35):

$$\left| m_0 \tau_0 g_{\mu\nu} \frac{dw^\mu}{d\tau} \frac{dw^\nu}{d\tau} \right|^{\mathrm{run-away}} < \frac{\tau_0}{m_0} \frac{e^2 c^2}{2\eta} \frac{|\eta f_0|}{(1 - \eta f_0)^2} + \frac{\tau_0}{m_0} \frac{2e^2 c^2}{7\eta} \frac{1}{1 - \eta f_0} \frac{(\eta f_0)^2}{(1 - \eta f_0)^2}$$

$$+ \frac{\tau_0}{m_0} \frac{\left| g_{\mu\nu} f_{\mathrm{ex}}{}^\mu f_{\mathrm{ex}}{}^\nu \right|}{(1 - \eta f_0)^2} + \frac{2\tau_0}{m_0} \frac{|\eta f_0|}{1 - \eta f_0} \frac{\times \left| g_{\mu\nu} f_{\mathrm{ex}}{}^\mu (^* f_{\mathrm{ex}})^\nu \right|}{(1 - \eta f_0)^2}$$

$$+ \frac{\tau_0}{m_0} \frac{\eta g_0}{(1 - \eta f_0)^2} \frac{\left| g_{\mu\nu} f_{\mathrm{ex}}{}^\mu (^* f_{\mathrm{ex}})^\nu \right|}{1 - \eta f_0} \tag{C4}$$

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